Towards the Determination of a Nonlinear Campbell Diagram of a Spinning Shaft with NonConstant Rotating Speed

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Abstract: In the literature it is reported only the studies for the steady states of spinning shafts without any information of the shaft's dynamics during spin-up, spin-down operation. The derivation of the system's discrete equations of motion is required for the analysis of the behavior of spinning shafts during this type of operation. The equations describing the motion of a spinning shaft with nonconstant rotating speed forms a system of nonlinear Partial Differential Equations (PDEs) coupled with an IntegroDifferential (ID) equation which describes the rigid body motion of the shaft. Towards the determination of the Nonlinear Campbell diagram of the spinning shaft the equations has been discretized by projecting the dynamics to the infinite base of the linear mode shapes of the underlying linear PDEs and the resulted discrete equations forms a nonlinear system in a Non-Cauchy format. Further on, based on the derived formulation the Campbell diagram has been determined by considering constant rotating speed and the results have been compared with those obtained by Finite Element analysis and they are in very good agreement. This work paves the way for the determination of the Nonlinear Campbell diagram by considering the periodic motions of the derived nonlinear discrete system using either numerical (shooting method with continuation) or analytical techniques. It should be mentioned that the same approach can be used also for spin-up, spin-down dynamic analysis with the determination of Nonlinear Campbell diagram in case of rotating blades with nonconstant rotating speed with application in turbomachinery, wind turbines, pumps and many other mechanical applications with rotating components.

***Keywords*:** Spinning shaft, Nonconstant rotating speed, Nonlinear Campbell diagram, modal coupling

1. Introduction

In the literature there are extensive dynamical studies of rotating and spinning mechanical components with constant rotating speed but very few considering nonconstant rotating speed. Therefore the dynamic analysis of the mechanical rotating components is limited only to steady states without having any information for the behavior of the systems during spin-up or spin-down operation. In prior of dynamic analysis during spin-up or spin down operation, there is the need to derive the equations of motion in a form that can be analyzed.

In [1] it has been derived the equations of motion of spinning shaft for nonconstant rotating speed but it was not considered the torsional motion and the resulted nonlinear IntegroDifferential (ID) equation for rigid body motion, involved only the nonlinear terms due to elastic motion in bending. In [2] it has been derived the equations of motion of a spinning shaft with nonconstant rotating speed and also considering eccentric sleeves as dynamic boundary conditions and these will be used herein. The derived system of equations is a nonlinear system of Partial Differential Equations (PDEs) and it should be mentioned that similar nonlinear inertia terms due to nonconstant rotating speed are present also in case of rotating blades as has been shown in [7].

In this article, in prior of moving to practical analysis it is necessary the system of PDEs to be discretized and it has been done by projecting the dynamics in the infinite basis of the modes of the underlying linear system. The discrete modal equations of motion have been derived, and they can be used for further nonlinear dynamic analysis. Also in prior of the nonlinear dynamic analysis, as intermediate step, the derived modal equations of motion are validated as follows. It is used the derived formulation and considering the particular case of constant rotating speed it has been determined the Campbell diagram. Then the Campbell diagram is compared with this obtained by using finite element formulation.

2. Analysis

We consider the equations of motion derived in [2] for linear Euler-Bernoulli beam, made of isotropic material, incorporating the rotary inertia terms and all nonlinear terms in rigid body motion but without the dynamic boundary conditions of the eccentric sleeves. The equations of motion are given by,

$\boldsymbol{\delta v}$ (**$\boldsymbol{v}$ bending motion in y-direction):** $\dot{\boldsymbol{\theta}}^2(\mathbf{I_1}\boldsymbol{v}) - \ddot{\boldsymbol{\theta}}(\mathbf{I_1}\boldsymbol{w}) - 2\dot{\boldsymbol{\theta}}(\mathbf{I_1}\dot{\boldsymbol{w}}) - (\mathrm{I}_1\ddot{v}) + (\mathrm{I}_5\ddot{v}')' - (k_5 v'')'' = 0,$ (1)

$\boldsymbol{\delta w}$ (**w bending motion in z-direction):** $\dot{\boldsymbol{\theta}}^2(\mathbf{I_1}\boldsymbol{w}) + \ddot{\boldsymbol{\theta}}(\mathbf{I_1}\boldsymbol{v}) + 2\dot{\boldsymbol{\theta}}(\mathbf{I_1}\dot{\boldsymbol{v}}) - (\mathrm{I}_1\ddot{w}) + (\mathrm{I}_6\ddot{w}')' - (k_6 w'')'' = 0,$(2)

$\boldsymbol{\delta\phi}$ (**$\boldsymbol{\phi}$ torsional motion):** $\dot{\boldsymbol{\theta}}^2(\mathbf{I_6}+\mathbf{I_5}) + \ddot{\theta}(\mathrm{I}_6+\mathrm{I}_5) - \left(\mathrm{I}_5\ddot{\phi} + \mathrm{I}_6\ddot{\phi}\right) + \left(\frac{k_7\phi'}{2} + \frac{k_8\phi'}{2}\right)' = 0,$ (3)

$\boldsymbol{\delta\theta}$ (**$\boldsymbol{\theta}$ rigid body motion of shaft):** $\ddot{\theta}(\mathrm{I}_5+\mathrm{I}_6)L + \ddot{\boldsymbol{\theta}}\int_0^L \mathbf{I_1}\boldsymbol{w}^2 + \mathbf{I_1}\boldsymbol{v}^2 + (\mathbf{I_5}+\mathbf{I_6})\,\boldsymbol{\phi}^2\boldsymbol{dx} =$

$$= \int_0^L (\mathrm{I}_5 + \mathrm{I}_6)\ddot{\phi}\,dx - \boldsymbol{2\dot{\theta} \int_0^L (\mathrm{I}_5 + \mathrm{I}_6)\dot{\phi}\phi + \mathrm{I}_1 v\dot{v} + \mathrm{I}_1 w\dot{w}dx} + \boldsymbol{\int_0^L \mathrm{I}_1 \ddot{w}v - \mathrm{I}_1 \ddot{v}w\,dx} \quad (4)$$

In bold are indicated the nonlinear terms, and these equations form a nonlinear system of PDEs coupled through the nonconstant rotating speed with one ID equation describing the rigid body motion of the shaft, and the following Boundary Conditions (B.C.s):

$$v(0,t) = v(L,t) = 0, \text{(5a,b)}, v''(0,t) = v''(L,t) = 0, \text{(6a,b)}, w(0,t) = w(L,t) = 0, \text{(7a,b)}$$
$$w''(0,t) = w''(L,t) = 0, \text{(8a,b)}, \phi(0,t) = 0, \text{(9)}\ \phi'(L,t) = 0. \text{(10)}$$

Whereas, eq. (5,7,9) are the strong B.C.s arising from the geometry of the problem and eq. (6,8,10) are the weak B.C.s arising from the equilibriums in free motions through the Extended Hamilton's Principle formulation. The coefficients are given by,

$$\mathrm{I}_1 = \pi\rho_0({r_o}^2 - {r_i}^2),\ \mathrm{I}_5 = \mathrm{I}_6 = \rho_0 I = \pi\rho_0\left(\frac{{r_o}^4 - {r_i}^4}{4}\right),\ k_5 = k_6 = EI,\ k_7 = k_8 = GI, \quad \text{(11a-d)}$$

with r_o, r_i, external and internal radius of shaft respectively, L is the length of the shaft, ρ_0, E, G are the density Young's and shear modulus respectively.

The system of equations (1-4) can be projected to the infinite base of the corresponding linear modes of the homogeneous linearized problem of these PDEs. The linearized system is forming decoupled PDEs whereas considering the B.C.s (eq. 5-10) it can be shown that the equations are also self-adjoint, therefore the linear mode shapes are orthogonal to each other and the linear modal equations decoupled.

The corresponding linear PDEs (of eq. 1,2) describing the motion in bending are identical for both directions and the associated Boundary Value Problem (BVP) solution can be shown that is having the following linear mode shapes and natural frequencies of the k -mode [3],

$$y_k(s) = \sqrt{\frac{2}{\mathrm{I}_1 L}}\sin\left(\frac{k\pi}{L}s\right),\quad \omega_k = \sqrt{\frac{k^4\pi^4 k_5}{L^2k^2\pi^2\mathrm{I}_5 + L^4\mathrm{I}_1}},\qquad k = 1,2,\ldots \quad (12,13)$$

respectively, whereas the mode shapes have been normalized considering only the I_1 terms.

The nonhomogeneous torsional BVP can be solved using the integral of the Green's function arising from the homogeneous problem multiplied with angular positions but they have to be defined explicitly in order to obtain specific solution. Therefore we restrict the solutions to the corresponding linear homogeneous PDE describing torsional motion which arise from equation 3 by neglecting terms associated with the rigid body motion and in this case the linear BVP is similar with those one for rod in axial vibration with clamped-free B.C.s. The solution of this BVP is having the following mode shapes and natural frequencies of the k -mode [4],

$$Y_k(s) = \sqrt{\frac{2}{(\mathrm{I}_6 + \mathrm{I}_5)L}}\sin\left(\frac{(2k-1)\pi}{2L}s\right),\quad \omega_{T,k} = \frac{(2k-1)\pi}{2L}\sqrt{\frac{(k_7 + k_8)}{2(\mathrm{I}_6 + \mathrm{I}_5)}},\quad k = 1,2,\ldots \quad (14,15)$$

respectively, whereas the mode shapes have been normalized.

The displacements in bending and the rotation due to torsion in equations (1,2,3) are expressed in series of the linear mode shapes as follows,

$$v = \sum_{k=1}^{n} y_k(s)\, q_{v,k}(t),\ w = \sum_{k=1}^{n} y_k(s)\, q_{w,k}(t),\ \phi = \sum_{k=1}^{n} Y_k(s)\, q_{\phi,k}(t), \quad \text{(16a,b,c)}$$

whereas, $q_{v,k}(t)$ is the k-mode modal displacement in y-direction of bending, $q_{w,k}(t)$ is the k-mode modal displacement in z-direction of bending, $q_{\phi,k}(t)$ is the k-mode modal displacement in torsion.

We multiply equations (1,2) with $y_j(s)$ and equation (3) with $Y_j(s)$, and we integrate over the length span of the shaft. As first attempt in order to simplify the problem we derived the discrete system of equations of motion with truncation of the series in first linear mode for each motion and also considering the rotating speed $\Omega = \dot{\theta}$ rather than the angular position θ of the rigid body motion of the shaft then the equations are taking the form,

-rigid body motion,

$$[(\mathrm{I}_5 + \mathrm{I}_6)L + q_v^2 + q_w^2 + q_\phi^2]\dot{\Omega} - \mathrm{F}\ddot{q}_\phi - q_v\ddot{q}_w + \ddot{q}_v q_w = -2\Omega\dot{q}_v q_v - 2\Omega\dot{q}_w q_w - 2\Omega\dot{q}_\phi q_\phi\,, \quad (17)$$

-modal equation in bending in y-direction, $\dot{\Omega}q_w + (1 - \mathrm{M})\ddot{q}_v = [\Omega^2 - \omega^2(1 - M)]q_v - 2\Omega\dot{q}_w,$ (18)

-modal equation in bending in z-direction, $-\dot{\Omega}q_v + (1 - \mathrm{M})\ddot{q}_w = [\Omega^2 - \omega^2(1 - M)]q_w + 2\Omega\dot{q}_v,$ (19)

-modal equation in torsion, $-\mathrm{F}\dot{\Omega} + \ddot{q}_\phi = F\Omega^2 - \omega_T^2 q_\phi,$ (20)

With constants,

$$\mathrm{F} = (\mathrm{I}_6 + \mathrm{I}_5)\int_0^L Y_1(s)ds = \frac{1}{\pi}\sqrt{(\mathrm{I}_6 + \mathrm{I}_5)8L},\qquad \mathrm{M} = \mathrm{I}_5\int_0^L {y_1}''(s)y_1(s)ds = -\frac{\mathrm{I}_5\pi^2}{\mathrm{I}_1 L^2}, \quad \text{(21a,b)}$$

3. Results

In order to validate the formulation and in prior of deriving the Nonlinear Campbell diagram by considering nonconstant rotating speed, it is considered the particular case of constant rotating speed ($\dot{\Omega} = 0$) which lead to the determination of the Campbell diagram. In this particular case, the equations (18-20) are taking the form,

-modal equation in bending in y-direction, $(1 - \mathrm{M})\ddot{q}_v - [\Omega^2 - \omega^2(1 - M)]q_v + 2\Omega\dot{q}_w = 0,$ (22)

-modal equation in bending in z-direction, $(1-\mathrm{M})\ddot{q}_w - [\Omega^2 - \omega^2(1-M)]q_w - 2\Omega\dot{q}_v = 0,$ (23)
-modal equation in torsion, $\ddot{q}_\phi + \omega_T^2 q_\phi = F\Omega^2.$ (24)

It should mentioned that only the two bending motions are coupled due to Coriolis force and the Campbell diagram can be derived by considering equations (22,23). Regarding modal equation 24 which describes torsional motion the centrifugal force is a constant (with respect to time) and fully decoupled from bending motion.

It can be shown, in case of Euler-Bernoulli beam (considering also the rotary inertia terms) that the Finite Element (FE) formulation is given by [6],

$$\begin{bmatrix} [M-M_I] & \mathbf{0} \\ \mathbf{0} & [M-M_I] \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & 2\Omega[M] \\ -2\Omega[M] & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} [K]-\Omega^2[M] & \mathbf{0} \\ \mathbf{0} & [K]-\Omega^2[M] \end{bmatrix} \begin{Bmatrix} q \\ p \end{Bmatrix} = 0, \quad (25)$$

whereas, $\ddot{q}, \dot{q}, q$, are the nodal acceleration,velocity,displacement respectively in y-direction, $\ddot{p}, \dot{p}, p$, are the nodal acceleration,velocity,displacement respectively in z-direction, Ω, is the rotating speed. The structural matrices of of the FE formulation arise using FE discretization of Euler Bernouli beam as it is shown in [8] with $[M]$, $[M_I]$, $[K]$ being the mass matrix, the matrix arising due to rotary inertia terms and the stiffness matrix respectively.

We consider a stainless steel shaft with, external $r_o = 0.031\,m$, and internal radius $r_i = 0.028\,m$, length of the shaft $L = 1.188\,m$, density $\rho_0 = 7850\,Kg/m^3$, and Young's modulus $E = 200\,GPa$. It should be mentioned that the particular shaft is thin-walled since the ratio of length with thickness is 396 (more than 10) therefore for the examination of the lower modes of vibration it can be modelled as Euler-Bernoulli beam by neglecting the shear [9].

The Campbell diagram for this shaft can be derived, by solving the eigenvalue problem. In case of the theoretical approach, it can be determined by considering equations (22,23) and also in case of FE formulation considering equation (25) both should be examined in state space form by considering a change of variables with new displacements a vector with displacements and velocities of the motions in equations and recasting the system (trivial procedure).

In Figure 1 are depicted the results for the determination of the Campbell diagram using the two approaches whereas the natural frequencies almost coincide and there is no significant difference between the results arising using the two different approaches. The results validate the derived theoretical formulation which can be used further on for nonlinear dynamic analysis.

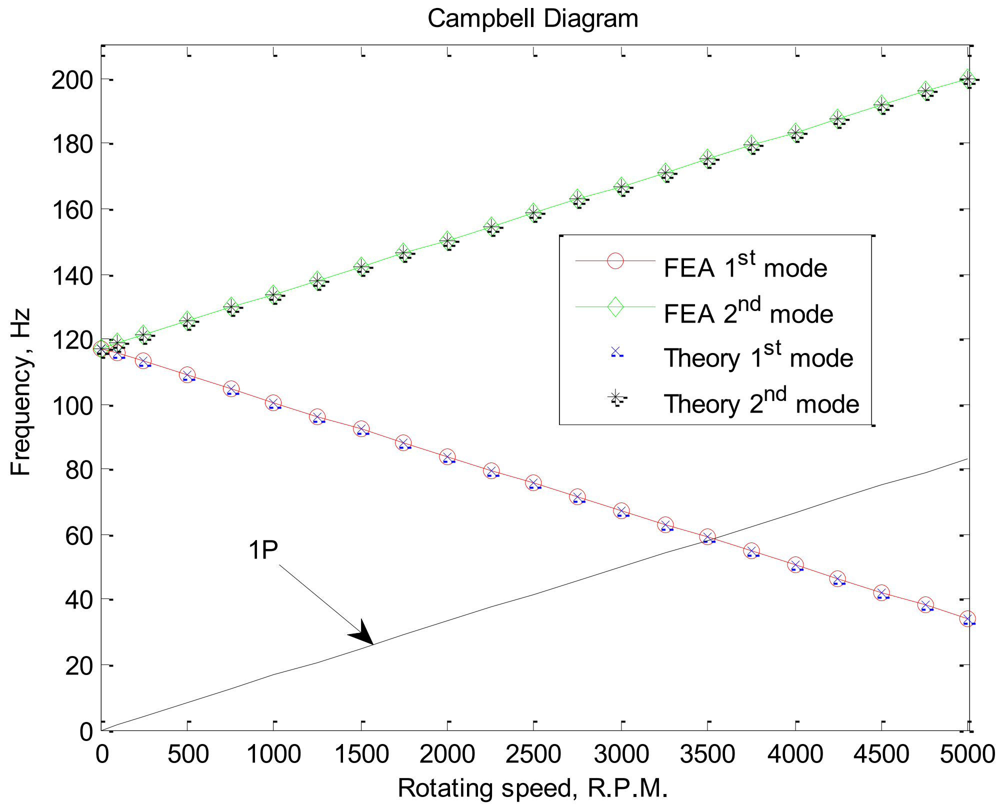

Figure 1. Campbell Diagram of spinning shaft.

4. Conclusion-Future work

Equations 17-20 forms a discrete nonlinear system of differential equations in Non-Cauchy form since involves nonlinearities in inertia terms. It should be mentioned that due to nonconstant rotating speed the torsional modal equation of motion is coupled with bending motions. Therefore nonlinear dynamic analysis of this system is expected to have modal interactions between bending and torsion, whereas in the case of constant rotating speed the modal coupling is restricted between bending motions due to Coriolis forces.

In prior of nonlinear dynamic analysis, it is considered the particular case of constant rotating speed in the modal equations and it is derived the Campbell diagram for a particular shaft by solving the eigenvalue problem. The validation of the formulation has been done with comparison of the results, with those obtained with finite element formulation, which are in very good agreement.

Towards the determination of the Nonlinear Campbell diagram of spinning shafts, it will be determined the periodic orbits of the nonlinear system formed by equations (17-20), using numerical (shooting method with a numerical continuation technique) and/or analytical techniques accompanied with numerical techniques. In case of numerical solution, the codes have been already developed by the author and they already used in other cases e.g. [5] with the only difference the existence of the state dependent mass matrix (system in Non-Cauchy form) which can be incorporated easily to the scheme through the multiplication of the vector field with the inverse of the mass matrix. Practically speaking the Nonlinear Campbell diagram corresponds to a broader case of the Campbell diagram derived with constant rotating speed and therefore the correlation of the two diagrams has to be examined.

Finally, the same approach can be used also in other cases, by means of the dynamic analysis during spin-up, spin-down with the derivation of the Nonlinear Campbell diagram of rotating blades with application in all rotating equipment e.g. turbomachinery, wind turbines etc.

5. References

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